

FOURIER SERIES

Trigonometric, Polar and Complex
Fourier Exponential Transforms

EEEN 462 – ANALOGUE COMMUNICATION SYSTEMS

Monday, 09 June 2025

WHERE ARE WE IN THE SYLLABUS?

Course content

Signal Analysis: Signal representation in time and frequency domain. Generalized expansion in complete orthonormal sets. Interpretation of signals as vectors in signal space.

Fourier series. Fourier and Laplace Transforms. Signal spectra and their properties. Rayleigh theorem. Sampling theorem for baseband and passband signals. Power and spectral energy densities. The DFT. the FFT. The Hilbert Transform. Deterministic signals through linear systems. Analog Modulation: Need for modulation. Linear Modulation: AM, DSB, VSB, SSB. Angle modulation FM & PM. Demodulators for AM and FM signals. Frequency division multiplexing. Noise in analog modulation: Effects of noise in AM and FM systems. Demodulator performance in the presence of noise. Pre-emphasis and de-emphasis filtering threshold in FM system. Comparison of system performance in noise (AM & FM).

WHAT IS FOURIER SERIES?

- **A Fourier series** is a way to represent a periodic function as a sum of sine and cosine functions **or equivalently**, as a sum of complex exponentials, each with different frequencies and amplitudes.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} a_n \sin nx$$

OR

- **A Fourier series** is an expansion of a periodic function $f(x)$ as an infinite sum of sines and cosines **or equivalently**, as a sum of complex exponentials, each with different frequencies and amplitudes.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{jnx}$$

CONDITIONS FOR EXISTANCE OF FOURIER SERIES

A **continuous-time Fourier series** exists for a function $x(t)$ when the following conditions (called the Dirichlet's conditions) are satisfied:

1. $x(t)$ is a well-defined single value function
2. $x(t)$ possesses a finite number of discontinuities in the period, T
3. $x(t)$ has a finite number of maxima and minima in the period, T

TROGONOMETRIC FOURIER SERIES

A periodic signal $x(t)$ can be expressed in the form of trigonometric Fourier series comprising the following sine and cosine terms:

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos(2\omega_0 t) + a_n \cos(n\omega_0 t) \\ + b_1 \sin \omega_0 t + b_2 \sin(2\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_0 n t) + b_n \sin(\omega_0 n t)$$

Where

$$t_0 < t < t_0 + T$$

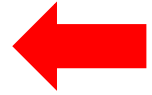
$$T = \frac{2\pi}{\omega_0}$$

ω_0 is the fundamental frequency and $2\omega_0$, $3\omega_0$ are the harmonics

TROGONOMETRIC FOURIER SERIES FOR PERIODIC FUNCTION

For a periodic function $x(t)$ with period T , we can write the Fourier coefficients as follows:

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$



The average value of the DC component of $x(t)$

$$\begin{aligned} a_n &= \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(\omega_0 n t) dt \\ &= \frac{2}{T} \int_0^T x(t) \cos(\omega_0 n t) dt \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(\omega_0 n t) dt \\ &= \frac{2}{T} \int_0^T x(t) \sin(\omega_0 n t) dt \end{aligned}$$

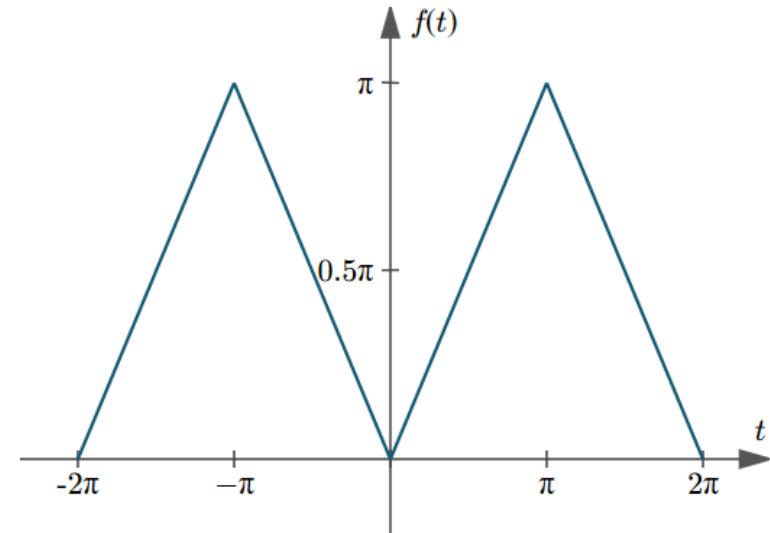
SYMMETRY CONDITIONS FOR INTEGRALS

Even Function

For even function,

$$x(-t) = x(t)$$

$$\int_{-a}^a x(t) dt = 2 \int_0^a x(t) dt$$



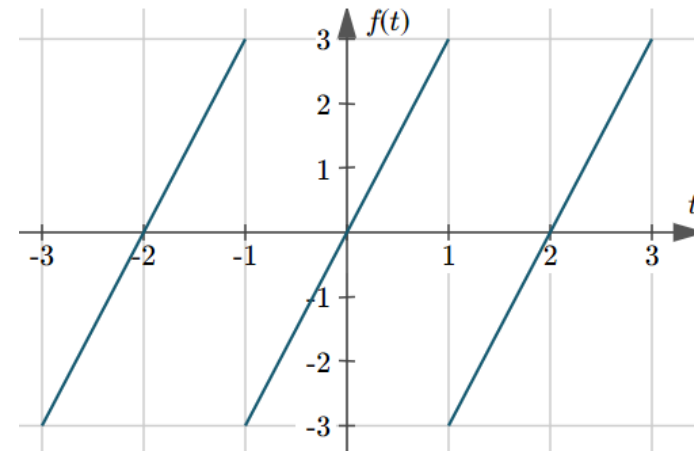
(a) Even triangular function

Odd Function

For odd function,

$$x(-t) = -x(t)$$

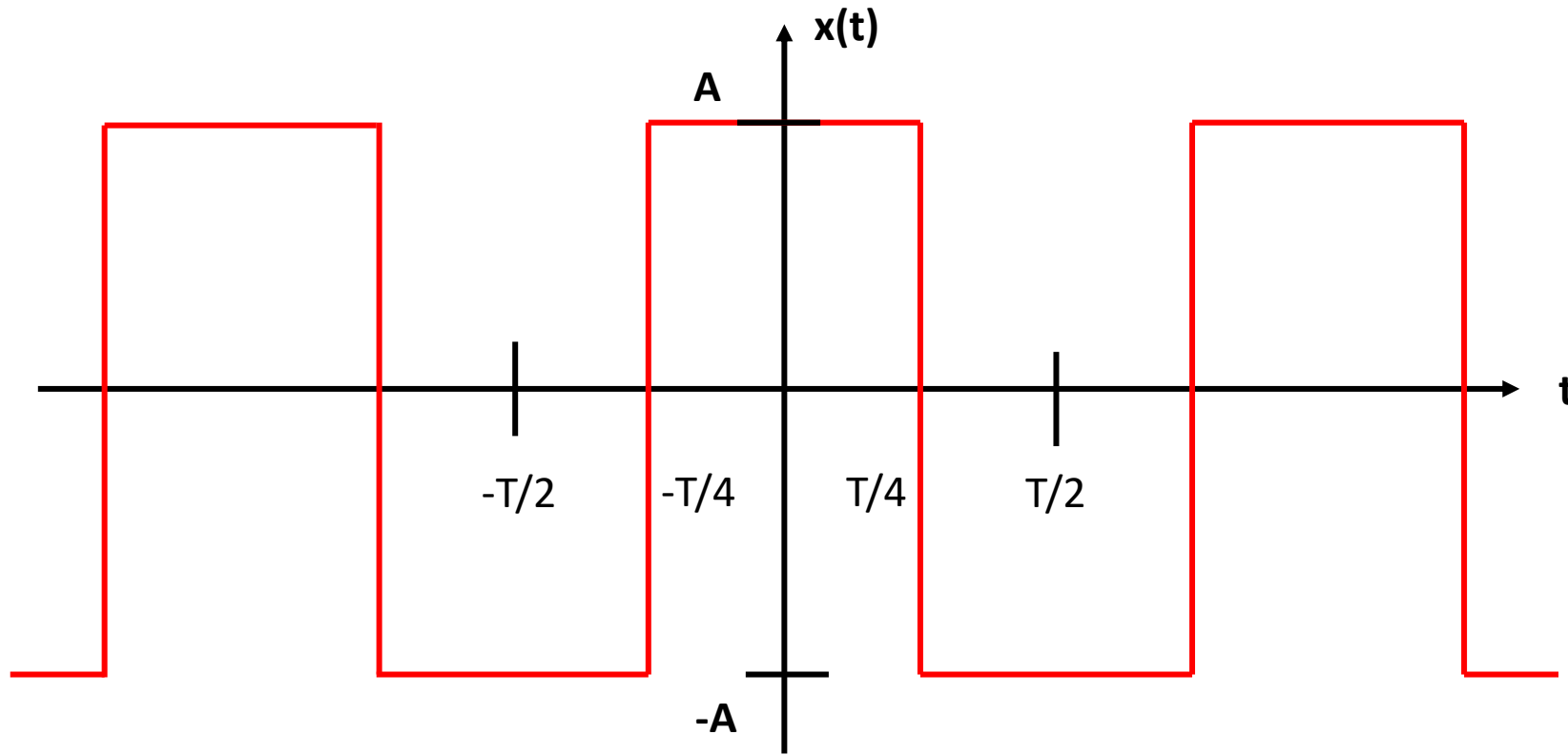
$$\int_{-a}^a x(t) dt = 0$$



(b) Odd sawtooth function

WORKED EXAMPLE -1

Obtain the Fourier series representation of a periodic square wave shown below.



WORKED EXAMPLE -1

The expression for Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_0 n t) + b_n \sin(\omega_0 n t)$$

1. Since the $x(t)$ is symmetrical about the x-axis, i.e $x(t) = -x(t)$, only the cosine terms are present.
2. The waveform is symmetrical about the horizontal axis, therefore the DC term $a_0 = 0$.

The expected Fourier series should therefore be of the form:

$$x(t) = \sum_{n=1}^{\infty} a_n \cos(\omega_0 n t)$$

We can write

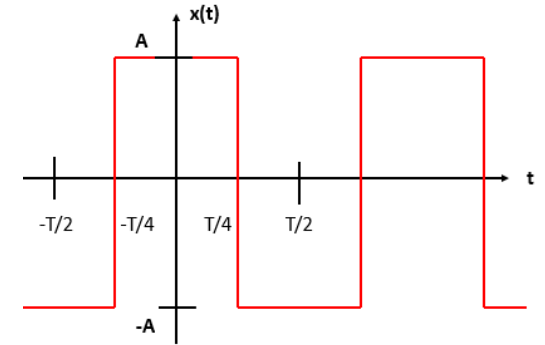
$$\begin{aligned} a_n &= \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos(\omega_0 n t) dt = \frac{2}{T} \int_{-T/4}^{T/4} A \cos(\omega_0 n t) + \frac{2}{T} \int_{-T/4}^{-T/2} -A \cos(\omega_0 n t) + \frac{2}{T} \int_{T/4}^{T/2} -A \cos(\omega_0 n t) \\ &= \frac{8A}{n\omega_0 T} \sin\left(\frac{n\omega_0 T}{4}\right) - \frac{4A}{n\omega_0 T} \sin\left(\frac{n\omega_0 T}{2}\right) \end{aligned}$$

Substituting $\omega_0 = \frac{2\pi}{T}$ the 2nd term becomes zero for all integer values of n , and we remain with

$$a_n = \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right) \text{ showing } a_n \text{ is non-zero for odd values of } n$$

or

$$x(t) = \frac{4A}{\pi} \left(\cos\omega_0 t - \frac{1}{3} \cos 3\omega_0 t + \frac{1}{5} \cos 5\omega_0 t + \dots \dots \right)$$



POLAR FOURIER SERIES REPRESENTATION

Polar Fourier representation is a modified form of trigonometric Fourier series which we have seen is given by:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_0 n t) + b_n \sin(\omega_0 n t)$$

or

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos(\omega_0 n t) + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin(\omega_0 n t) \right]$$

If

$$\frac{a_n}{\sqrt{a_n^2 + b_n^2}} = \cos \varphi_n \text{ and } \frac{b_n}{\sqrt{a_n^2 + b_n^2}} = \sin \varphi_n \text{ So that } \tan \varphi_n = \frac{b_n}{a_n}$$

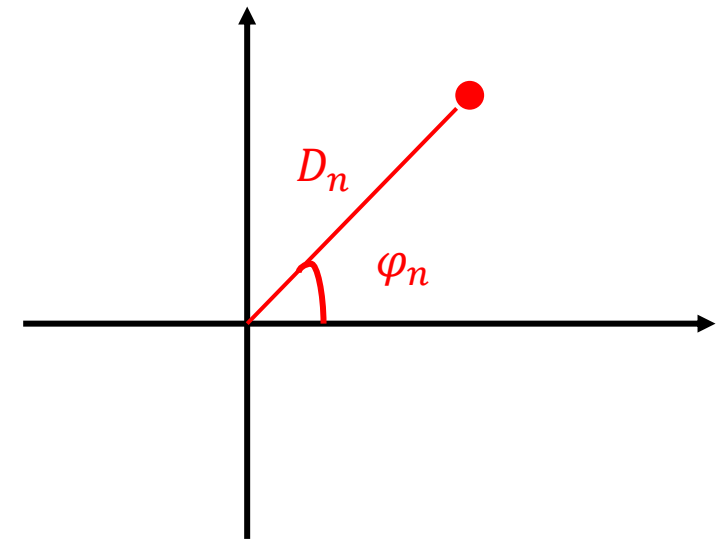
We can then write

$$x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} (\cos(\omega_0 n t - \varphi_n))$$

where

$D_n = \sqrt{a_n^2 + b_n^2}$ is referred to as spectral amplitude of the function $x(t)$

φ_n is referred to as the phase spectrum of the function $x(t)$



EXPONENTIAL FORM FOURIER SERIES /01

The complex form of Fourier series is simpler and more compact. It is therefore more widely used in signal analysis.

From trigonometric Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega_0 nt) + b_n \sin(\omega_0 nt)$$

We know from Euler Identity that

$$e^{i\theta} = \cos\theta + j\sin\theta \quad \text{or} \quad e^{-i\theta} = \cos\theta - j\sin\theta$$

We can write

$$\begin{aligned} \cos \omega_0 nt &= \frac{1}{2} (e^{j\omega_0 nt} + e^{-j\omega_0 nt}) \\ \sin \omega_0 nt &= \frac{1}{2} (e^{j\omega_0 t} - e^{-j\omega_0 t}) \end{aligned}$$

Substituting, we get

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} (e^{j\omega_0 nt} + e^{-j\omega_0 nt}) + \frac{b_n}{2} (e^{j\omega_0 t} - e^{-j\omega_0 t})$$

EXPONENTIAL FORM FOURIER SERIES /02

$$x(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} (e^{j\omega_0 nt} + e^{-j\omega_0 nt}) + \frac{b_n}{2} (e^{j\omega_0 t} - e^{-j\omega_0 t})$$

We can write

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n e^{j\omega_0 nt} + \sum_{n=-\infty}^{-1} C_{-n} e^{-j\omega_0 nt}$$

Which can be rewritten as:

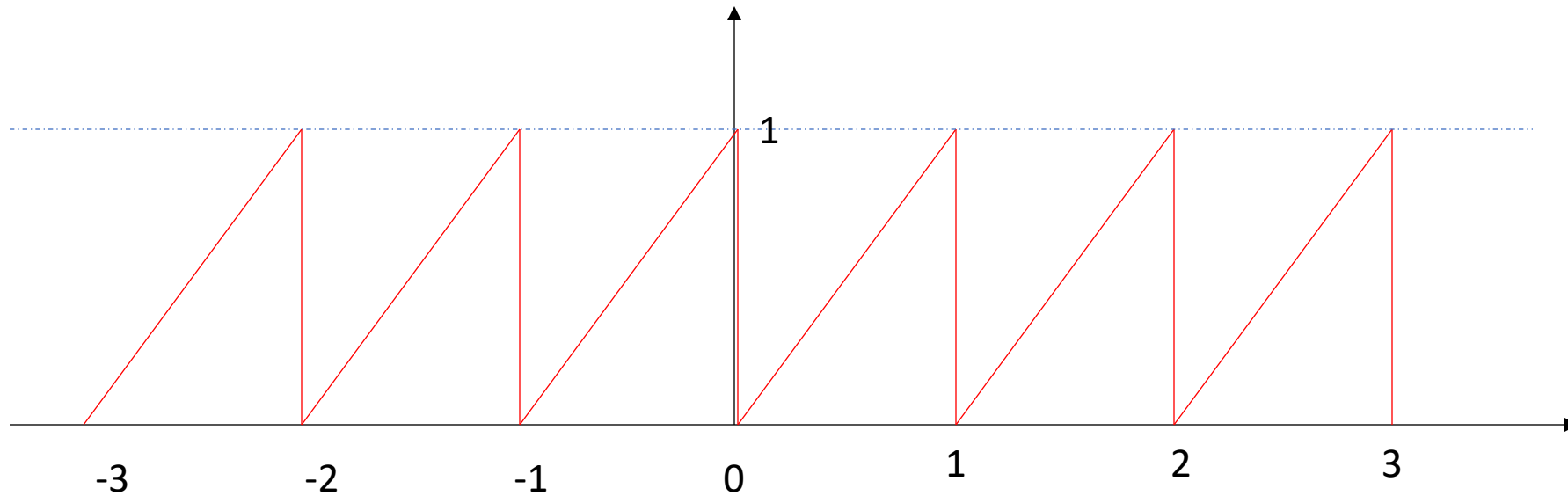
$$x(t) = c_0 + \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 nt}$$

Where

$$c_n = \frac{1}{T} \int_{-\pi/2}^{\pi/2} x(t) e^{-j\omega_0 nt} dt$$

WORKED EXAMPLE

Find the exponential-form Fourier series of the sawtooth waveform shown below. Plot the magnitude and phase.



WORKED EXAMPLE / 01

Over one period, the signal may be represented as:

$$x(t) = t \quad \text{for } 0 < t < 1 \quad \text{and the period } T = 1$$

The Fourier series coefficients of the signal $x(t)$ are therefore

$$C_n = \frac{1}{T} \int_0^T t e^{-i2\pi nt/T} dt = \int_0^1 t e^{-i2\pi nt} dt$$

Using the integration identity

$$\int x e^{ax} dx = e^{ax} \left(\frac{x}{a} - \frac{1}{a^2} \right)$$

Taking $a = -j2\pi n$ we get

$$C_n = \frac{-j}{2\pi n} \quad \text{for } n \neq 0$$

And

$$C_0 = \int_0^T x(t) dt = \int_0^1 t dt = \frac{1}{2}$$

WORKED EXAMPLE / 02

For $n \neq 0$

$$|C_n| = \sqrt{0 + \frac{1}{(2\pi n)^2}} = \frac{1}{2\pi n}$$

For $n = 0$

$$C_0 = \frac{1}{2}$$

The phase

$$\theta_n = \arg(C_n) = \pm 90^\circ$$

